Response to Anonymous Referee #2

We are very grateful to reviewer for careful and friendly review of our paper. All the comments are very useful and helped us to improve the manuscript.

The following corrections were made:

P.1, Line 14: "Over years the problem" should be "Over the years, the problem of…"

Corrected.

P.1, Line 15: " the process of dissipation which takes place" should be "….that takes place".

Corrected.

P.2, line 1: drop 'topical'.

Done.

P.2, line 11 g is the gravitational acceleration, reads better.

Corrected.

P.2, line 22 – there are inconsistencies in the definition of the stability parameter. L, the Obukhov length, is capital whereas z/l is used throughout – it should be z/L. Same issue on p.3, line 2.

Corrected.

The coefficient Cu in equation (8) …

The difference is due to definition of the Obukhov length-scale. We define it as $L = \frac{\tau^{3/2}}{-\beta F_z}$, while in papers on the Kansas experiment it is defined including the von Karman constant: $L = \frac{\tau^{3/2}}{-\beta F_z k}$. This just makes approximately two times difference in empirical constants.

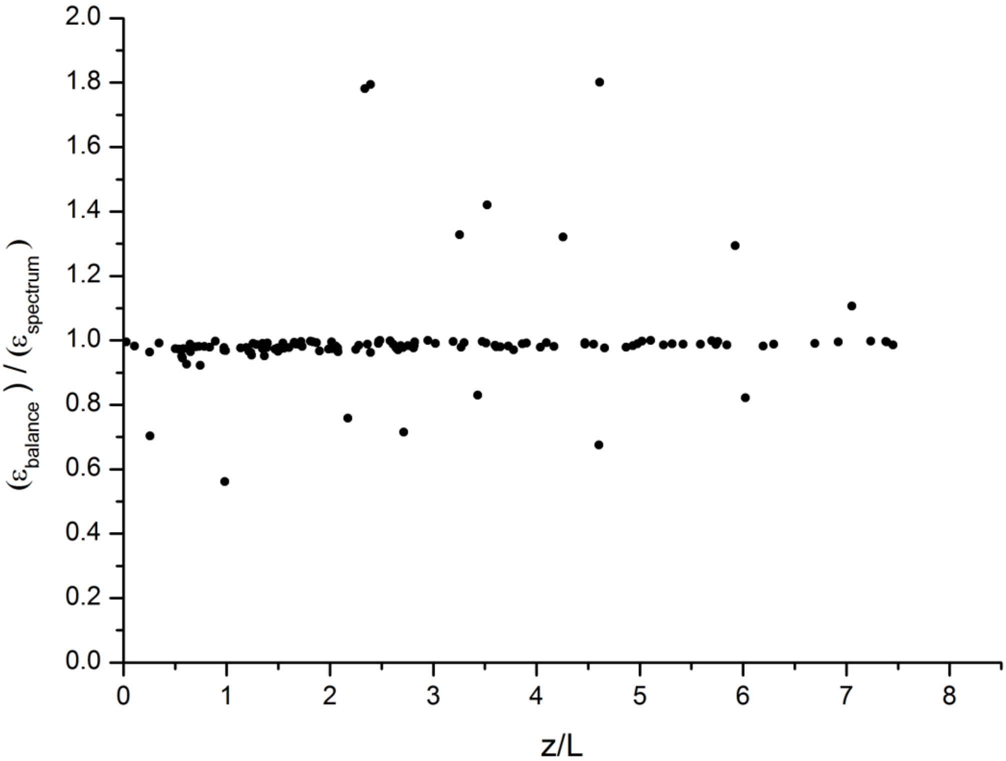
The value of R_∞ …

P.4, line 6, the following sentences were added:

It is worth noting that R_∞ can be derived from well-established phenomenological constants of turbulence characterising the inertial subrange (Katul et al., 2014). The actual value in this case is slightly higher ($R_\infty = 0.25$) but still within reasonable range.

Page 4, lines 27-28: I think it is worth showing a 1:1 comparison of the mean turbulent kinetic energy dissipation rate estimates from the spectrum and from the residual of the TKE budget.

Such comparison surely deserves consideration. Different types of data used in our paper show very good correspondence between the dissipation rates estimated (i) from spectrum and (ii) as the residual of TKE budget. The difference is within 5% (see the figure below).



In this paper we relay basically on DNS and use atmospheric data only to illustrate principal agreement between surface-layer data and DNS data. We would not like to include additional figures for the following reason. Additional (and not very necessary) figures inevitably diverge the attention of readers from the basic subject.

Page 6, Equation (20) is really the main result as it shows how the turbulent potential energy and the turbulent kinetic energy play a role in shaping the mean turbulent kinetic energy dissipation rate with stability. May be worth expanding this connection in the conclusion

Thank you for this valuable remark. We add the following sentences.

1) In the very end of section 3:

There is essential advantage of Ri_E as criterion of stratification in numerical modelling. Turbulent fluxes are usually calculated through familiar *diagnostic* down-gradient formulations: $\boldsymbol{\tau} = -K_M\, \partial \mathbf{U}/\partial z$ and $F_z = -K_H\, \partial \Theta/\partial z$, where K_M is eddy viscosity and K_H is eddy conductivity. Then, finite-difference approximation of the gradients causes uncertainties in $\boldsymbol{\tau}$, F_z and, hence, the Obukhov length, L [Eq. (3)], flux Richardson number, Ri_f [Eq. (5)], and gradient Richardson number, Ri [Eq. (4)]. Contrastingly, TKE and TPE are defined from *prognostic* budget equations accounting for turbulent diffusion which smooths the energies and assures quite certain calculation of Ri_E.

2) In the very end of concluding remarks:

Universal analytical formulation of $\left(kz/\tau^{1/2}\right)\partial U/\partial z$ versus z/L yields the single-valued relations linking z/L as criterion of stratification in the surface-layer flow or $\tilde{z}/L$ as the same criterion in Couette flow with alternative criterions: flux Richardson number, Ri_f [Eq. (5)], and the newly introduced "energy Richardson number", Ri_E [Eq. (13)], applicable to any turbulent regimes. This opens prospects for extending the obtained dependence of dissipation rate on static stability to any stably stratified turbulent flows.

Figure 3 – worth adding the best-fit line from the Kansas data as well.

Same reasoning applies here as in comment on comparison of dissipation rate estimated from spectrum and as the residual of TKE budget: in this paper we relay basically on DNS and use atmospheric data only to illustrate principal agreement between surface-layer data and DNS data.

Response to Anonymous Referee #3

Thank you for careful reviewing and friendly comments. We fully agree that readers must have clear info of DNS and observational data used in our paper. Luckily, a new paper presenting detailed info about our DNS is already in press (Mortikov et al., in press). Thanks to your comment, we are including the proper reference to the manuscript.

We recall that our paper focuses on new knowledge of dissipation rate in stably stratified turbulence. In this context, we do not write much about DNS and observational data as such. We do that on purpose. Otherwise some readers would overlook factual essence of our paper and comprehend it as just an empirical validation of known results. We bear in mind the interests of the majority of readers forced to look through plenty of publications.

Moreover, we think that introduction of detailed description of DNS and atmospheric data into the manuscript would, first, duplicate the info that can be easily found in already published papers (referenced in our manuscript and available to readers just by click) and, second, diverge reader's attention from the basic subject matter. Instead of repeating the already published info, we give references. The accuracy of the data and possible errors are considered in corresponding papers.

The following text has been added on page 4, line 11:

To assure accuracy of numerical simulations, we employed two DNS codes: INM-RAS, and IAP-RAS. Despite using two different codes developed separately and characterised by different spatial and temporal schemes, resolutions and statistical averaging the two types of DNS have shown quite consistent results that can be considered as a cross-validation. For detailed description of our numerical models used see Mortikov (2016), Mortikov et al. (in press) and Druzhinin et al. (2016).

The reference has been added on page 9, line 21:

Mortikov E.V., Glazunov A.V., Lykosov V.N.: Numerical study of plane Couette flow: turbulence statistics and the structure of pressure-strain correlations, Russian Journal of Numerical Analysis and Mathematical Modelling, 34, 2, 2019 (in press).

Dissipation rate of turbulent kinetic energy in stably stratified sheared flows

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Abstract. Over the years, the problem of dissipation rate of turbulent kinetic energy (TKE) in stable stratification remained unclear because of the practical impossibility to directly measure the process of dissipation that takes place at the smallest scales of turbulent motions. Poor representation of dissipation causes intolerable uncertainties in turbulence-closure theory and, thus, in modelling stably stratified turbulent flows. We obtain theoretical solution to this problem for the whole range of stratifications from neutral to limiting stable; and validate it via (i) direct numerical simulation (DNS) immediately detecting the dissipation rate and (ii) indirect estimates of dissipation rate retrieved via the TKE-budget equation from atmospheric measurements of other components of the TKE-budget. The proposed formulation of dissipation rate will be of use in any turbulence-closure models employing the TKE budget equation and in problems requiring precise knowledge of the high-frequency part of turbulence spectra in atmospheric chemistry, aerosol science and microphysics of clouds.

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1 Introduction

Until present, the dependence of dissipation rate, ε_K, of turbulent kinetic energy (TKE), E_K, on static stability remained insufficiently understood that caused principal difficulties in turbulence-energetics/closure theory and intolerable uncertainties in comprehending and modelling stably-stratified turbulent flows. Traditionally, the dissipation rate is parameterised in terms of a turbulent length-scale, l_T, as $\varepsilon_K \sim E_K^{3/2}/l_T$. This solves the problem in neutrally stratified boundary-layer flow, when the only length-scale is the distance over the surface, z, so that $l_T \sim z$. However, in stratified flows, one more length-scale appears, namely the Obukhov length scale, L, so that the ratio l_T/z becomes an unknown function of z/L. To define this function we combine observational evidence with theoretical analyses. We employ the steady-state TKE budget equation to retrieve data on dissipation versus stability from uncountable data on wind profiles in

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the moderately stably stratified atmospheric surface layers; supplement this information with our own direct numerical simulation of turbulence in stably stratified Couette flow; and combine the collected empirical knowledge with asymptotic analysis of the TKE equation. The analyses reveals perfect equivalence of our *asymptotic formulation of the velocity profile in extremely stable stratification and well-known log-linear velocity profile in moderately stable stratifications* typical of the atmospheric surface layer – up to the coincidence of empirical dimensionless constants. This very lucky empirical finding yields universal formulation of the dissipation rate versus static stability, valid over the whole range of stratifications from neutral to extremely stable. The formulation is applicable to any stationary and horizontally homogeneous stably stratified sheared flows and can be used within any turbulence-closure model equipped with TKE budget equation.

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For certainty, we consider the dissipation rate of TKE in terms of dry atmosphere, where fluctuation of buoyancy, $b = \beta\theta$, is proportional to fluctuation of potential temperature, θ; $\beta = g/T_0$ is the buoyancy parameter; g is the gravitational acceleration; and T_0 is reference value of absolute temperature. Since Kolmogorov (1942), ε_K is expressed through the dissipation length-scale l_T or time-scale, t_T:

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$$\varepsilon_K = \frac{E_K}{t_T} = \frac{E_K^{3/2}}{l_T}. \tag{1}$$

This formulation is not hypothetical but just defines the scales t_T and l_T, so that Eq. (1) merely expresses one unknown, ε_K, through another, t_T or l_T. In neutrally stratified boundary-layer flows, the only principal length-scale is the height over the surface, z; so that l_T is proportional to z, which yields

$$l_T = C_l z, \quad \varepsilon_K = \frac{E_K^{3/2}}{C_l z}, \tag{2}$$

where C_l is dimensionless constant to be determined empirically.

Stratification involves the Obukhov length-scale:

$$L = \frac{\tau^{3/2}}{-\beta F_z}, \tag{3}$$

where τ is absolute value of vertical turbulent flux of momentum $\boldsymbol{\tau} = (\tau, 0)$, and F_z is vertical turbulent flux of potential temperature (Obukhov, 1946). The restraining effect of stable stratification on turbulence is characterised by the dimensionless height, z/L; gradient Richardson number:

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$$\mathrm{Ri} = \frac{\beta\,\partial\Theta/\partial z}{(\partial \mathbf{U}/\partial z)^2}; \tag{4}$$

or flux Richardson number:

$$\mathrm{Ri}_f = \frac{\beta F_z}{\boldsymbol{\tau}\cdot\partial \mathbf{U}/\partial z}, \tag{5}$$

where $\partial \mathbf{U}/\partial z$ and $\partial \Theta/\partial z$ are vertical gradients of mean wind velocity, $\mathbf{U} = (U, V)$, and mean potential temperature, Θ. Then the dimensionless dissipation rate, $\varepsilon_K z/E_K^{3/2}$, is no longer a constant but depends on stratification (z/L, Ri or Ri_f). Until recently, practically nothing was known about this dependence beyond the interval of stratifications covered by observations in atmospheric surface layer: $0 < \mathrm{Ri} < 0.2$, which corresponds to $0 < z/L < 10$.

2 Dissipation rate in the steady-state stably stratified sheared flows

We consider horizontally homogeneous stationary boundary-layer flow in semi-space $z > 0$ as idealised model of atmospheric surface layer. Here, the familiar TKE budget equation expresses the dissipation rate, ε_K, through τ, F_z and $\partial U/\partial z$:

$$\varepsilon_K = \tau \frac{\partial U}{\partial z} + \beta F_z = \tau \frac{\partial U}{\partial z}\left(1 - \mathrm{Ri}_f\right). \qquad (6)$$

With increasing static stability, Ri_f obviously increases but (because $\varepsilon_T > 0$) remains limited, which is why it must tend to a finite limit: $\mathrm{Ri}_f \to R_\infty < 1$. Atmospheric data and results from direct numerical simulation (DNS) demonstrated below in Figures 2 and 3 confirm such behaviour and yield quite certain estimate of $R_\infty = 0.2$.

Then, substituting R_∞ for Ri_f in Eq. (5) yields asymptotic expression of the velocity gradient in extremely stable stratification:

$$\frac{\partial U}{\partial z} \to \frac{1}{R_\infty}\frac{\tau^{1/2}}{L} \equiv \frac{\tau^{1/2}}{L}\left(\frac{k}{R_\infty}\frac{z}{L}\right) \text{ at } \frac{z}{L} \to \infty. \qquad (7)$$

Here, the von Karman constant, k, is inserted in numerator and denominator just to highlight consistency of Eq. (7) with well-known formulation of the velocity-gradient in weakly and moderately stable stratifications typical of atmospheric surface layers obtained in the familiar Monin-Obukhov Similarity Theory (MOST):

$$\frac{\partial U}{\partial z} = \frac{\tau^{1/2}}{kz}\left(1 + C_u \frac{z}{L}\right), \qquad (8)$$

where $C_u = 2$ is well-established dimensionless empirical constant (Monin and Obukhov, 1954, Monin and Yaglom, 1971, Garratt, 1992, Stull, 1997). Originally, Eq. (8) was derived as the first term in the Taylor expansion of the dimensionless velocity gradient, $\Phi_M = (kz/\tau^{1/2})\partial U/\partial z$, considered in MOST as universal function of z/L. Subsequently, it was revealed that Eq. (8) with $C_u = 2$ is valid over the whole range of z/L observed in atmospheric surface layers, $0 < z/L < 10$, which corresponds to quite small gradient Richardson numbers: $0 < \mathrm{Ri} < 0.2$ (Monin and Yaglom, 1971). By this means, Eq. (8) with $C_u = 2$ based on massive atmospheric data for moderately stable stratifications yields at $z/L \to \infty$ precisely the same limit as Eq. (7) with $k/R_\infty = 2$ resulted from the conventional value of $k = 0.4$ and new estimate of the critical flux Richardson number: $R_\infty = 0.2$ obtained from topical DNS and available atmospheric data for maximal stable stratifications (Figure 2). This lucky coincidence just means that Eq. (8) with $C_u = k/R_\infty$ hold true in any stable stratification:

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$$\frac{\partial U}{\partial z} = \frac{\tau^{1/2}}{kz}\left(1 + \frac{k}{R_\infty}\frac{z}{L}\right) \text{ at } 0 < z/L < \infty\,. \quad (9)$$

Then, substituting Eq. (9) for $\partial U/\partial z$ into Eqs. (5) and (6), yields the following simple relations linking Ri_f with z/L:

$$\mathrm{Ri}_f = \frac{kz/L}{1+kR_\infty^{-1}z/L}\,,\ \frac{z}{L} = \frac{R_\infty}{k}\frac{\mathrm{Ri}_f}{R_\infty - \mathrm{Ri}_f}\,, \quad (10)$$

and exact formulation of the TKE dissipation rate as dependent on static stability:

$$\varepsilon_K = \frac{\tau^{3/2}}{kz}\left[1 + k\left(R_\infty^{-1} - 1\right)\frac{z}{L}\right] = \frac{\tau^{3/2}}{kz}\frac{1-\mathrm{Ri}_f}{1-\mathrm{Ri}_f/R_\infty}\,. \quad (11)$$

It is worth noting that R_∞ can be derived from well-established phenomenological constants of turbulence characterising the inertial subrange (Katul et al., 2014). The actual value in this case is slightly higher ($R_\infty = 0.25$) but still within a reasonable range.

To comprehensively validate the above analyses, we performed direct numerical simulation (DNS) of stably stratified *Couette flow*, namely, the plain-parallel flow between two horizontal plates separated in the vertical by the distance d, and moving with constant velocity in opposite directions. To assure accuracy of numerical simulations, we employed two DNS codes: INM-RAS, and IAP-RAS. Despite using two different codes developed separately and characterised by different spatial and temporal schemes, resolutions and statistical averaging the two types of DNS have shown quite consistent results that can be considered as a cross-validation. For detailed description of our numerical models used see Mortikov (2016), Mortikov et al. (in press) and Druzhinin et al. (2016).

Deleted: different codes: INM-RAS (Mortikov, 2016), and IAP-RAS (Druzhinin et al., 2016) that have shown quite consistent results .

In our DNS total (turbulent + molecular) fluxes of momentum, τ, and potential temperature, F_z, are practically equal to the turbulent fluxes elsewhere beyond narrow near-wall sublayers where molecular transports dominate. In the Couette flow the total fluxes are constant with height, exactly as in the surface-layer flows. Similarly, the flux-profile relations linking τ and F_z with vertical gradients of mean velocity, $\partial U/\partial z$, and potential temperature, $\partial \Theta/\partial z$, as well as the budget equations for turbulent energies, in particular Eq. (6), are the same as in the surface-layer flows. The only difference is in geometry of domains illustrated in Figure 1.

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Following Obukhov (1942), we distinguish between "absolute geometry" characterised by the usual height over the surface, z, and "internal geometry" characterised in Couette flow by specific vertical coordinate, $\tilde{z}$, dictated by conformal mapping of the Couette-flow domain $(0 < z < d)$ into the semi-space:

$$\tilde{z} = \frac{d}{\pi}\sin\frac{\pi z}{d} \text{ in Couette flow } (0 < z < d)\,. \quad (12)$$

This coordinate reflects equal influences of the lower and upper walls on the fluid flow.

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In semi-space, the "internal geometry" coincides with "absolute geometry": $\tilde{z} = z$. Thus, the vertical structure of the Couette flow in terms of $\tilde{z}$ coincides with the vertical structure of the surface-layer flow in terms of z. This allows showing in the same framework the genuine dissipation rate calculated from DNS: $\varepsilon_K = \nu\langle(\partial u_i/\partial x_k)(\partial u_i/\partial x_k)\rangle$, where ν is kinematic

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viscosity, together with $\varepsilon_K = \tau\,\partial U/\partial z + \beta F_z$ retrieved from atmospheric observations assuming the steady-state TKE budget.

In Figures 2-4 we show our DNS data together with atmospheric data on the dissipation rate retrieved from observations in the surface layers indirectly:

- via the Kolmogorov –5/3 power law from measured spectra of TKE in the inertial subrange (Pearson et al., 2002), and
- via the steady-state TKE budget Eq. (6) from the measured turbulent fluxes of momentum, τ, and potential temperature, F_z, and vertical gradient of mean wind velocity, $\partial U/\partial z$.

In these figures DNS data are shown by heavy coloured dots; and atmospheric data, by light grey symbols.

Figure 2 shows flux Richardson number, $\mathrm{Ri}_f = \beta F_z \left(\boldsymbol{\tau} \cdot \partial \mathbf{U}/\partial z\right)^{-1}$, versus dimensionless height, $\tilde{z}/L$, in Couette flow; or versus z/L in atmospheric surface layer. The black curve is plotted after Eq. (10) taking conventional value of the von Karman constant: $k = 0.4$ and our estimate of the maximal flux Richardson number: $R_\infty = 0.2$ resulted from best fit of Eq. (10) to DNS data. Notably, total (turbulent + molecular) fluxes of momentum, τ, and potential temperature, F_z, in Couette flow are constant across the flow which assures very certain specification of Ri_f and L, and makes our DNS most suitable for calibrating the theory. We recall that Eqs. (10) and (11) are relevant to the well-developed turbulence regime where molecular transports are negligible, so that turbulent fluxes practically coincide with total fluxes. In our DNS, it is so except for the narrow transition layers dominated by molecular transport near the lower and upper walls: $0 < \tilde{z} < 50\nu/\tau^{1/2}$. Data from these layers are indicated by dark grey *points*. The light grey *symbols* show atmospheric data from the following sources: research observatory Tiksi in East Siberia near the Arctic Ocean coast (Grachev et al., 2018); offshore oceanographic platform in the Black Sea (Repina et al., 2009); and acoustic soundings over arid-steppe in the Republic of Kalmykia in Southern Russia (Vazaeva et al., 2017). In spite of inevitable heterogeneity, non-stationarity and other side effects, atmospheric data correlate quite well with DNS data.

Figure 3 shows dimensionless dissipation rate, $\varepsilon_K z/\tau^{3/2}$, versus z/L after Eq. (11) and atmospheric data; and $\varepsilon_K \tilde{z}/\tau^{3/2}$ versus $\tilde{z}/L$ after DNS in Couette flow. All notations are the same as in Figure 2. The theoretical curve plotted after Eq. (11) with $k = 0.4$ and $R_\infty = 0.2$ is fully consistent with experimental data, except for the narrow transition layer $0 < \tilde{z} < 50\nu/\tau^{1/2}$ where Eq. (11) is irrelevant. Hence, Figure 3 justifies the stability dependence of dissipation rate, Eq. (11), and provides additional confirmation to the empirical estimate of $R_\infty = 0.2$.

In Figure 2 we use as argument the flux Richardson number: $\mathrm{Ri}_f = \frac{\beta F_z}{\boldsymbol{\tau} \cdot \partial \mathbf{U}/\partial z}$, where turbulent fluxes (disregarding molecular contributions in the transition layer) appear in both numerator and denominator. Hence uncertainties in both fluxes are somehow compensated. This is not the case in Figure 3 showing $\varepsilon_K z/\tau^{3/2}$ vs. z/L: the dissipation rate in the numerator is just total dissipation, whereas the momentum flux in denominator disregards the molecular contribution. This causes the ugly looking but only natural dark gray points on the left side of Figure 3.

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3 Turbulent length-scales and general criterion of stratification

The concept of the TKE dissipation rate directly relates to definition of the turbulent time-scale, $t_T \equiv E_k/\varepsilon_K$, and length-scale, $l_T \equiv E_K^{1/2} t_T = E_K^{3/2}/\varepsilon_K$. Then Eq. (11) defines l_T as function of z/L:

$$l_T \equiv E_K^{1/2} t_T = \frac{E_K^{3/2}}{\varepsilon_K} = kz\left(\frac{E_K}{\tau}\right)^{3/2}\left[1 + k\left(R_\infty^{-1} - 1\right)\frac{z}{L}\right]^{-1} . \qquad (13)$$

It has the asymptotic limits:

$$l_T \to k\left(E_K/\tau\right)_0^{3/2} z \sim z \text{ at } z/L \to 0 \text{ and } l_T \to \frac{R_\infty}{1-R_\infty}\left(\frac{E_K}{\tau}\right)_\infty^{3/2} L \sim L \text{ at } z/L \to \infty , \qquad (14)$$

where the limits of E_K/τ in neutral stratification and in extremely stable stratification are just dimensionless constants. Our DNS yield the following estimates: $\left(E_K/\tau\right)_0 \approx 4$ and $\left(E_K/\tau\right)_\infty \approx 11$. The length-scale similar to Eq. (13) was already revealed as inherent to spectra of turbulence in unstably stratified boundary-layer flows (Glazunov, 2014).

We emphasise that l_T is the scalar characterising turbulence as a whole. Contrastingly, turbulent mixing in different directions is characterised by the mixing-lengths vector $l_{Ti} \equiv E_{Ki}^{1/2} t_T$ $(i = 1,2,3)$ with generally different stream-wise $(i = 1)$, transverse $(i = 2)$ and vertical $(i = 3)$ components. We emphasise principal difference between scalar *length-scale* and vector *mixing-length*. In literature, the words "turbulent length-scale" and "turbulent mixing-length" are often used as interchangeable. This cause intolerable confusion because different components of the mixing length differently depend on static stability (Zilitinkevich et al., 2013).

The above analyses are done for the simplest surface-layer (or Couette) flow, where dimensionless height z/L (or $\tilde{z}/L$) plays the role of criterion quantifying the effect of stratification on turbulence. Luckily, our major result [Eqs. (11) and (13)] can be easily extended to a wide range of stratified turbulent flows. We recall that stratified turbulence is characterised, besides E_K, by turbulent potential energy (TPE), $E_P = \frac{1}{2}\beta\langle\theta^2\rangle/\partial\Theta/\partial z$. Hence the effect of stratification on turbulence can be quantified by the "energy Richardson number" defined as

$$\mathrm{Ri}_E = \frac{E_p}{E_K} . \qquad (15)$$

In contrast to traditional criteria, such as Ri (Eq. 4), Ri_f (Eq. 5), or z/L, the energy Richardson number criterion is valid in heterogeneous and non-stationary flows, for any mechanisms of generation of turbulence (including breaking waves, oscillating grid, etc.) and in flows with complex geometry.

Expressing the dissipation rates of TKE and TPE in the steady state through the dissipation time scale, $l_T \equiv E_K^{3/2}/\varepsilon_K$, the budget equations for TKE and TPE become

$$E_K = t_T\left(-\boldsymbol{\tau} \cdot \partial \mathbf{U}/\partial z + \beta F_z\right) , \qquad (16)$$

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$$E_P = \frac{1}{2}\frac{\beta\langle\theta^2\rangle}{\partial\Theta/\partial z} = -C_P t_T \beta F_z \,, \tag{17}$$

where C_P is dimensionless parameter quantifying the difference between the dissipation rates of TKE and TPE (Zilitinkevich et al., 2013). Equations (16) and (17) in combination with Eq. (10) yield the following relations linking Ri_E with Ri_f or z/L:

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$$\mathrm{Ri}_E = \frac{C_p}{\mathrm{Ri}_f^{-1}-1} = \frac{C_P kz/L}{1+\left(R_\infty^{-1}-1\right)kz/L} \,. \tag{18}$$

Figure 4 shows Ri_E versus z/L or $\tilde{z}/L$ (like in previous figures) after our DNS and atmospheric observations. Theoretical curve is plotted after Eq. (18) taking $k = 0.4$, $R_\infty = 0.2$ and empirical estimate of the dimensionless parameter $C_P = 0.62$ just obtained from the best fit of Eq. (18) to DNS data. Experimental data reveal the asymptotic limit:

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$$\mathrm{Ri}_E \to R_{E\infty} = \frac{C_p}{R_\infty^{-1}-1} = 0.155 \ \text{ at } \ z/L \to \infty \,, \tag{19}$$

Then, using Eq. (18) to express z/L through Ri_E, Eq. (11) in terms of Ri_E becomes:

$$\varepsilon_K = \varepsilon_{K(neutral)}\left(1 - \frac{\mathrm{Ri}_E}{R_{E\infty}}\right)^{-1} , \tag{20}$$

where $\varepsilon_{K(neutral)}$ is dissipation rate in neutral stratification. In the surface layer $\varepsilon_{K(neutral)} = \tau^{3/2}/kz$; but generally $\varepsilon_{K(neutral)}$ depends on concrete energy-generation mechanisms and geometry of flow.

There is essential advantage of Ri_E as criterion of stratification in numerical modelling. Turbulent fluxes are usually calculated through familiar *diagnostic* down-gradient formulations: $\boldsymbol{\tau} = -K_M\, \partial\mathbf{U}/\partial z$ and $F_z = -K_H\, \partial\Theta/\partial z$, where K_M is eddy viscosity and K_H is eddy conductivity. Then, finite-difference approximation of the gradients causes uncertainties in $\boldsymbol{\tau}$ and F_z and, hence, in the Obukhov length, L [Eq. (3)], flux Richardson number, Ri_f [Eq. (5)], and gradient Richardson number, Ri [Eq. (4)]. Contrastingly, TKE and TPE are defined from *prognostic* budget equations accounting for turbulent diffusion which smooths the energies and assures quite certain calculation of Ri_E.

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4 Concluding remarks

The dissipation rate of TKE, ε_K, as dependent on static stability over years remained uncertain because of impossibility of direct measurement of ε_K. Admittedly, ε_K can be retrieved via the TKE budget equation from the measured turbulent fluxes, τ and F_z, and mean-velocity gradient, $\partial U/\partial z$, and also via the Kolmogorov –5/3 power law from the measured spectra of TKE in the inertial subrange. However, these methods are justified only in stationary and horizontally homogeneous flows and require fully controlled conditions. These provisions, practically unachievable in atmospheric experiments, make estimates of ε_K from atmospheric observations rather uncertain. Wide spread of atmospheric data is clearly seen in our figures. Moreover, available atmospheric data cover only weakly to moderately stable stratifications typical of the surface layer. To avoid these difficulties, we performed topical DNS of the steady-state, stably stratified turbulent Couette flows up

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to the strongest attainable stratifications; combined direct data from DNS with data retrieved from atmospheric observations; and employed theoretical analysis to reveal asymptotic behaviour of the mean velocity gradient and the dissipation rate in extremely stable stratification, namely, at $z/L \to \infty$, where L is the Obukhov length-scale.

By providential coincidence, the formulations happen to be precisely the same in the asymptotic limit $z/L \to \infty$ and in the weakly stable stratifications $0 < z/L < 10$ typical of atmospheric surface layer. This yields simple analytical formulations of the dimensionless velocity gradient, $(kz/\tau^{1/2})\,\partial U/\partial z$, and dissipation rate, $(kz/\tau^{3/2})\varepsilon_K$, as universal functions of z/L [Eq. (9) and Eq. (11)] across the whole range of stratifications from neutral to extremely stable.

Universal analytical formulation of $(kz/\tau^{1/2})\,\partial U/\partial z$ versus z/L yields the single-valued relations linking z/L as criterion of stratification in the surface-layer flow or $\tilde{z}/L$ as the same criterion in Couette flow with alternative criterions: flux Richardson number, Ri_f [Eq. (5)], and the newly introduced "energy Richardson number", Ri_E [Eq. (13)], applicable to any turbulent regimes. This opens prospects for extending the obtained dependence of dissipation rate on static stability to any stably stratified turbulent flows.

6 Author contribution

Sergej Zilitinkevich invented the concept, did theoretical derivations and led the research. Evgeny Mortikov (author of INM-RAS DNS code), and Oleg Druzhinin (author of IAP-RAS DNS code) carried out independent numerical simulation of turbulent Couette flow aiming at verification and validation of the theory. Irina Repina collected and analysed complementary data from observations in atmospheric surface layer over various types of the Earth surface. Evgeny Kadantsev utilised all these data, carried out the entire work on verification and validation, and produced all figures. Yulia Troitskaya and Andrey Glazunov contributed to theoretical analyses. All authors participated in writing the manuscript.

7 Competing interests

The authors declare that they have no conflict of interest.

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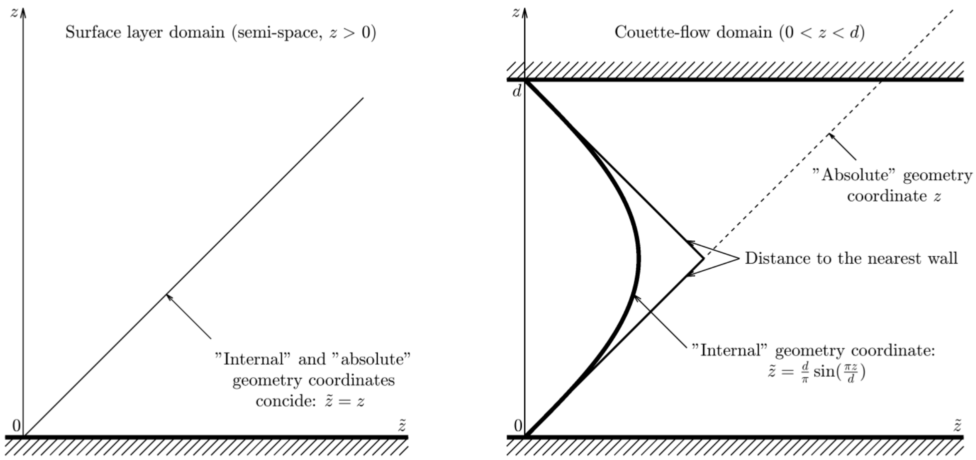

Figure 1: Usual height, z, and vertical coordinate, $\tilde{z}$, defined by Eq. (12) characterising "absolute" and "internal" geometry of the domain, respectively. Left panel shows semi-space, $z > 0$, where $\tilde{z} = z$. Right panel shows the layer between two horizontal walls, $0 < z < d$, where $\tilde{z}$ coincides with the distance from nearest surface in its close vicinity, but essentially depends on the distances from both surfaces in the central part of the domain.

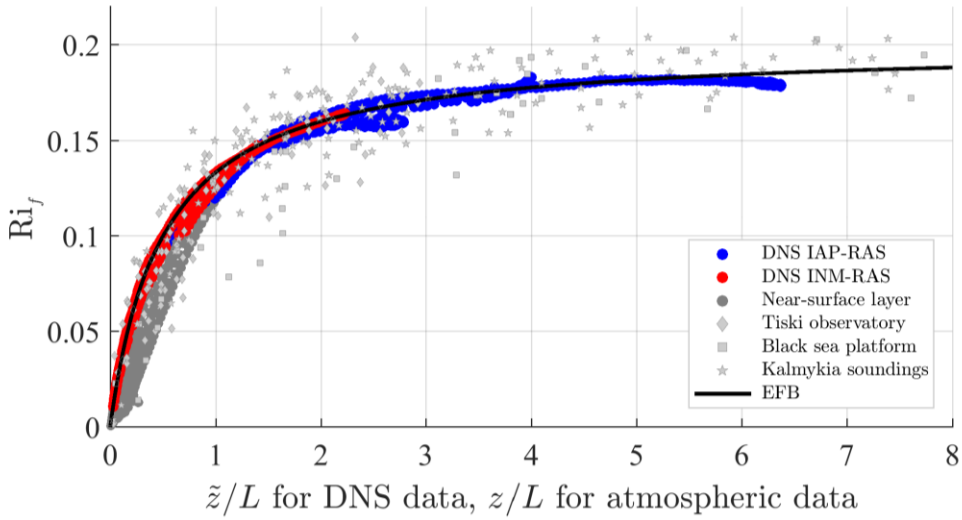

Figure 2: Flux Richardson number, Ri_f, in stable stratification versus $\tilde{z}/L$ in Couette flow or versus z/L in atmospheric surface layer. Empirical data used for the calibration are obtained in two series of DNS runs employing INM-RAS code (red dots) and IAP-RAS code (blue dots). Atmospheric data are taken from Arctic coastal observatory Tiski (light grey diamonds), Black sea offshore platform (light grey squares) and acoustic soundings in Kalmykia steppe (light grey stars). Dark grey dots belong to very narrow near-surface layer: $0 < \tilde{z} < 50\nu/\tau^{1/2}$. Black solid line shows Eq. (11) with conventional value of the von-Karman constant, $k = 0.4$, and new empirical vale of $R_\infty = 0.2$ just obtained from the best fit of Eq. (10) to DNS data from elsewhere beyond the layer $0 < \tilde{z} < 50\nu/\tau^{1/2}$, where molecular transports are significant and Eq. (10) is not necessarily relevant.

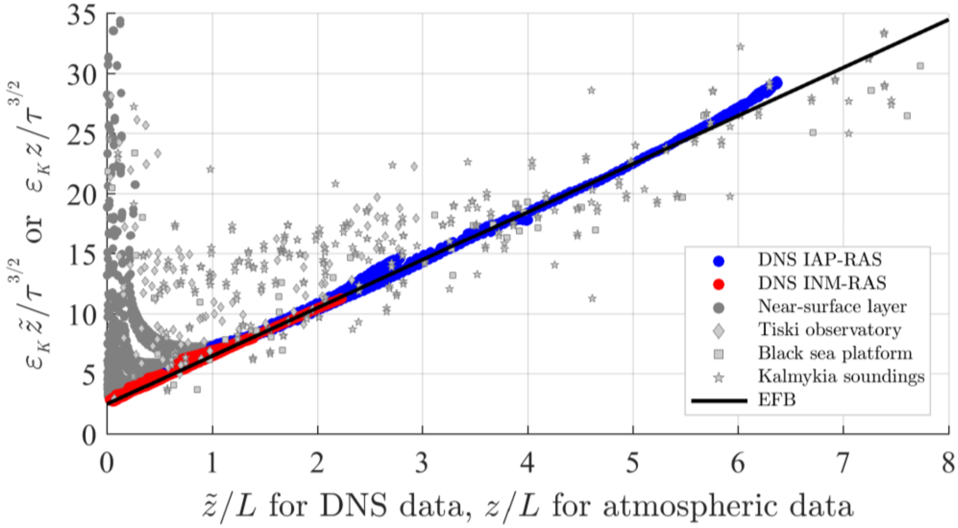

Figure 3: Dimensionless dissipation rate in stable stratification, $\varepsilon_K \tilde{z}/\tau^{3/2}$, versus $\tilde{z}/L$ in Couette flow or $\varepsilon_K z/\tau^{3/2}$ versus z/L in atmospheric surface layer. Empirical data are from the same sources as in Figure 2. Black solid line shows Eq. (11) with $k = 0.4$ and $R_\infty = 0.2$. Dark grey dots belong to the very narrow near-surface layer: $0 < \tilde{z} < 50\nu/\tau^{1/2}$, where molecular transports are significant and Eq. (11) is not relevant.

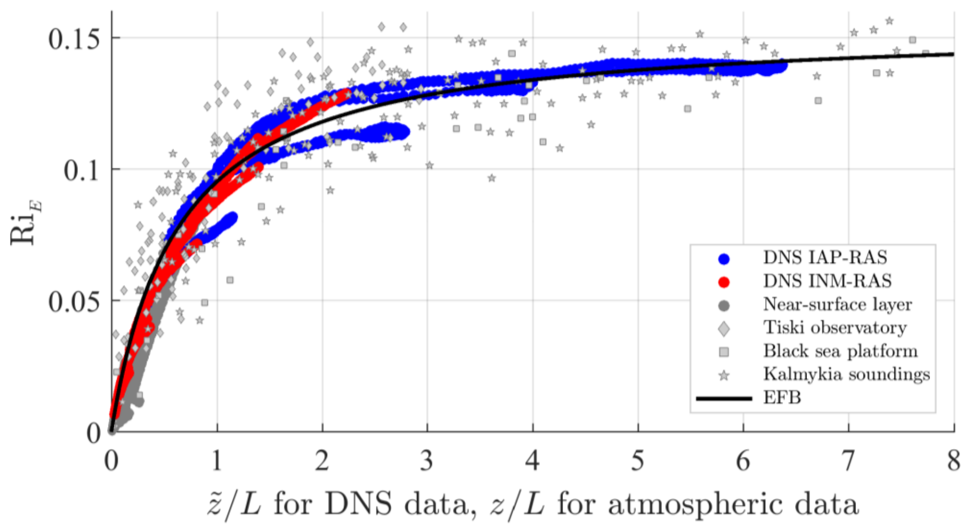

Figure 4: The energy Richardson number, $\mathrm{Ri}_E = E_p/E_K$, versus $\tilde{z}/L$ in Couette flow or versus z/L in atmospheric surface layer. Empirical data are from the same sources as in Figures 2 and 3. Black solid line shows Eq. (16) with $k = 0.4$, $R_\infty = 0.2$, and $C_P = 0.62$.

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